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Computation of inviscid compressible flows with the V-SGS stabilization and $YZ\beta$ shock-capturing

Franco Rispoli[1], Rafael Saavedra[2], Alessandro Corsini[1,*,†] and Tayfun E. Tezduyar[3]

[1] *Dipartimento di Meccanica e Aeronautica, Università degli Studi di Roma 'La Sapienza', Via Eudossiana, 18, I-00184 Roma, Italy*
[2] *Departamento de Ingeneria Mecánica-Electrica, Universidad de Piura, Av. Ramón Mugica 131 Piura, Perú*
[3] *Mechanical Engineering, Rice University—MS 321, 6100 Main Street, Houston, TX 77005, U.S.A.*

SUMMARY

The $YZ\beta$ shock-capturing technique was introduced recently for use in combination with the streamline-upwind/Petrov–Galerkin formulation of compressible flows in conservation variables. The $YZ\beta$ shock-capturing parameter is much simpler than an earlier parameter derived from the entropy variables for use in conservation variables. In this paper, we propose to use the $YZ\beta$ shock-capturing in combination with the variable subgrid scale (V-SGS) formulation of compressible flows in conservation variables. The V-SGS method is based on an approximation of the class of SGS models derived from the Hughes variational multiscale method. We evaluate the performance of the V-SGS and $YZ\beta$ combination in a number of standard, 2D test problems. Compared to the earlier shock-capturing parameter derived from the entropy variables, in addition to being much simpler, the $YZ\beta$ shock-capturing parameter yields better shock quality in these test problems. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The streamline-upwind/Petrov–Galerkin (SUPG) formulation of compressible flows is one of the earliest stabilized finite element formulations. It was first introduced in 1982, soon after the

*Correspondence to: Alessandro Corsini, Dipartimento di Meccanica e Aeronautica, Università degli Studi di Roma 'La Sapienza', Via Eudossiana, 18, I-00184 Roma, Italy.
†E-mail: corsini@dma.ing.uniroma1.it, alessandro.corsini@uniroma1.it

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introduction of the SUPG formulation of incompressible flows [1, 2]. This first SUPG formulation of compressible flows was in the context of conservation variables, and was described in detail in a NASA technical report [3]. A concise version of the NASA report was published as an AIAA paper [4], and a more thorough version with additional examples as a journal paper [5]. That SUPG formulation did not involve any shock-capturing term. It was later recast in entropy variables and supplemented with a shock-capturing term [6]. In a 1991 ASME paper [7], the SUPG formulation introduced in [3–5] was supplemented with a very similar shock-capturing term, which included a shock-capturing parameter that is now called 'δ_{91}'. This shock-capturing parameter was derived from the one given in [6] for the entropy variables. With the test computations reported in [7, 8], it was shown that with this shock-capturing term, the SUPG formulation introduced in [3–5] is very comparable in accuracy to the one that was recast in entropy variables.

New ways of calculating the shock-capturing parameters to be used with the SUPG formulation of compressible flows in conservation variables were introduced in [9, 10]. The new shock-capturing parameters, which we now categorize as '$YZ\beta$ shock-capturing', are simpler and less costly to compute with than δ_{91}. They are based on scaled residuals and are defined with options for smoother or sharper shocks. A preliminary set of test computations with these new shock-capturing parameters were reported in [11] for inviscid supersonic flows. Those computations were limited to very simple 2D geometries and quadrilateral elements. A more comprehensive set of 2D test computations for inviscid supersonic flows were reported in [12]. Those tests with the $YZ\beta$ shock-capturing involved different element types and mesh orientations. In [13], numerical experiments were carried out for inviscid supersonic flows around cylinders and spheres to evaluate the performance of the $YZ\beta$ shock-capturing in more challenging test problems. In those numerical experiments, in addition to comparing the $YZ\beta$ results to those obtained with δ_{91}, for 2D structured meshes, the $YZ\beta$ results were compared to the results obtained with the OVERFLOW code [14].

The variable subgrid scale (V-SGS) method was first introduced in [15] for the advection–diffusion–reaction equation and for incompressible flows. The V-SGS method is based on an approximation of the class of SGS models derived from the Hughes variational multiscale (Hughes-VMS) method [16]. In [17], the V-SGS approach was formulated for compressible flows in conservation variables.

In this paper, we propose a method where the $YZ\beta$ shock-capturing is used in combination with the V-SGS formulation of compressible flows in conservation variables. We evaluate the performance of the V-SGS and $YZ\beta$ combination in a number of standard, 2D test problems. In Section 2, we review the governing equations of compressible flows in conservation variables. The SUPG and V-SGS formulations are described in Section 3, and the $YZ\beta$ shock-capturing in Section 4. The test computations are presented in Section 5, and the concluding remarks are given in Section 6.

2. NAVIER–STOKES EQUATIONS OF COMPRESSIBLE FLOWS

Let $\Omega \subset \mathbb{R}^{n_{sd}}$ be the spatial domain with boundary Γ, and $(0, T)$ be the time domain. The symbols ρ, $\mathbf{u}$, p and e will represent the density, velocity, pressure and the total energy, respectively.

The Navier–Stokes equations of compressible flows can be written on Ω and $\forall t \in (0, T)$ as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} - \frac{\partial \mathbf{E}_i}{\partial x_i} - \mathbf{R} = \mathbf{0} \tag{1}$$

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where $\mathbf{U} = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho e)^{\mathrm{T}}$ is the vector of conservation variables, and $\mathbf{F}_i$ and $\mathbf{E}_i$ are, respectively, the Euler and viscous flux vectors

$$\mathbf{F}_i = \begin{pmatrix} u_i \rho \\ u_i \rho u_1 + \delta_{i1} p \\ u_i \rho u_2 + \delta_{i2} p \\ u_i \rho u_3 + \delta_{i3} p \\ u_i(\rho e + p) \end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix} 0 \\ T_{i1} \\ T_{i2} \\ T_{i3} \\ -q_i + T_{ik} u_k \end{pmatrix} \tag{2}$$

Here δ_{ij} are the components of the identity tensor $\mathbf{I}$, q_i are the components of the heat flux vector, and T_{ij} are the components of the Newtonian viscous stress tensor

$$\mathbf{T} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) \tag{3}$$

where λ and μ $(= \rho \nu)$ are the viscosity coefficients, ν is the kinematic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain-rate tensor

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \tfrac{1}{2}((\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathrm{T}}) \tag{4}$$

It is assumed that $\lambda = -2\mu/3$. The equation of state used here corresponds to the ideal gas assumption. The term $\mathbf{R}$ represents all other components that might enter the equations, including the external forces.

Equation (1) can further be written in the following form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij} \frac{\partial \mathbf{U}}{\partial x_j}\right) - \mathbf{R} = \mathbf{0} \tag{5}$$

where

$$\mathbf{A}_i = \frac{\partial \mathbf{F}_i}{\partial \mathbf{U}}, \quad \mathbf{K}_{ij} \frac{\partial \mathbf{U}}{\partial x_j} = \mathbf{E}_i \tag{6}$$

Appropriate sets of boundary and initial conditions are assumed to accompany Equation (5).

3. SUPG AND V-SGS STABILIZATIONS

In describing the SUPG and V-SGS formulations of Equation (5), we assume that we have constructed some suitably defined finite-dimensional trial solution and test function spaces $\mathscr{S}^h_{\mathbf{U}}$ and $\mathscr{V}^h_{\mathbf{U}}$. Based on that, the SUPG [4, 5, 7] and V-SGS [17] formulations can be written as follows: find $\mathbf{U}^h \in \mathscr{S}^h_{\mathbf{U}}$ such that $\forall \mathbf{W}^h \in \mathscr{V}^h_{\mathbf{U}}$

$$\int_\Omega \mathbf{W}^h \cdot \left(\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}^h_i \frac{\partial \mathbf{U}^h}{\partial x_i}\right) \mathrm{d}\Omega + \int_\Omega \left(\frac{\partial \mathbf{W}^h}{\partial x_i}\right) \cdot \left(\mathbf{K}^h_{ij} \frac{\partial \mathbf{U}^h}{\partial x_j}\right) \mathrm{d}\Omega$$

$$- \int_{\Gamma_{\mathrm{H}}} \mathbf{W}^h \cdot \mathbf{H}^h \,\mathrm{d}\Gamma - \int_\Omega \mathbf{W}^h \cdot \mathbf{R}^h \,\mathrm{d}\Omega$$

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$$+\sum_{e=1}^{n_{\mathrm{el}}}\int_{\Omega^e}\mathbf{P}_{\mathrm{STAB}}(\mathbf{W}^h)\cdot\left[\frac{\partial\mathbf{U}^h}{\partial t}+\mathbf{A}_i^h\frac{\partial\mathbf{U}^h}{\partial x_i}-\frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij}^h\frac{\partial\mathbf{U}^h}{\partial x_j}\right)-\mathbf{R}^h\right]\mathrm{d}\Omega$$

$$+\sum_{e=1}^{n_{\mathrm{el}}}\int_{\Omega^e}\nu_{\mathrm{SHOC}}\left(\frac{\partial\mathbf{W}^h}{\partial x_i}\right)\cdot\left(\frac{\partial\mathbf{U}^h}{\partial x_i}\right)\mathrm{d}\Omega=0 \tag{7}$$

where $\mathbf{H}^h$ represents the natural boundary conditions associated with Equation (5), and Γ_{H} is the part of the boundary where such boundary conditions are specified. The vector operator $\mathbf{P}_{\mathrm{STAB}}(\mathbf{W}^h)$ takes the following forms for the SUPG and V-SGS stabilizations, respectively

$$\mathbf{P}_{\mathrm{STAB}}(\mathbf{W}^h)=\mathbf{P}_{\mathrm{SUPG}}(\mathbf{W}^h) \tag{8}$$

$$\mathbf{P}_{\mathrm{STAB}}(\mathbf{W}^h)=\mathbf{P}_{\mathrm{VSGS}}(\mathbf{W}^h) \tag{9}$$

where

$$\mathbf{P}_{\mathrm{SUPG}}(\mathbf{W}^h)=\left[\boldsymbol{\tau}_{\mathrm{SUPG}}\left(\frac{\partial\mathbf{W}^h}{\partial x_k}\right)\right]\mathbf{A}_k^h \tag{10}$$

$$\mathbf{P}_{\mathrm{VSGS}}(\mathbf{W}^h)=\left[(\mathbf{A}_k^h)^{\mathrm{T}}\left(\frac{\partial\mathbf{W}^h}{\partial x_k}\right)+\frac{\partial}{\partial x_l}\left((\mathbf{K}_{lk}^h)^{\mathrm{T}}\left(\frac{\partial\mathbf{W}^h}{\partial x_k}\right)\right)\right]\boldsymbol{\tau}_{\mathrm{VSGS}} \tag{11}$$

The diagonal matrices $\boldsymbol{\tau}_{\mathrm{SUPG}}$ and $\boldsymbol{\tau}_{\mathrm{VSGS}}$ are the SUPG and V-SGS stabilization parameters. The expressions for these matrices can be found in [9–13] for the SUPG stabilization and in [17] for the V-SGS stabilization. The shock-capturing parameter is denoted by ν_{SHOC}. It was discussed briefly in Section 1 and will further be discussed in Section 4.

4. $YZ\beta$ SHOCK-CAPTURING

In the 'YZ' version of the $YZ\beta$ shock-capturing, ν_{SHOC} is defined as

$$\nu_{\mathrm{SHOC}}=\|\mathbf{Y}^{-1}\mathbf{Z}\|\left(\sum_{i=1}^{n_{\mathrm{sd}}}\left\|\mathbf{Y}^{-1}\frac{\partial\mathbf{U}^h}{\partial x_i}\right\|^2\right)^{\beta/2-1}\left(\frac{h_{\mathrm{SHOC}}}{2}\right)^{\beta} \tag{12}$$

Here $\mathbf{Y}$ is a diagonal scaling matrix constructed from the reference values of the components of $\mathbf{U}$

$$\mathbf{Y}=\begin{bmatrix}(U_1)_{\mathrm{ref}} & 0 & 0 & 0 & 0\\ 0 & (U_2)_{\mathrm{ref}} & 0 & 0 & 0\\ 0 & 0 & (U_3)_{\mathrm{ref}} & 0 & 0\\ 0 & 0 & 0 & (U_4)_{\mathrm{ref}} & 0\\ 0 & 0 & 0 & 0 & (U_5)_{\mathrm{ref}}\end{bmatrix} \tag{13}$$

$$\mathbf{Z}=\mathbf{A}_i^h\frac{\partial\mathbf{U}^h}{\partial x_i} \tag{14}$$

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and

$$h_{\mathrm{SHOC}} = h_{\mathrm{JGN}} \tag{15}$$

where

$$h_{\mathrm{JGN}} = 2\left(\sum_{a=1}^{n_{\mathrm{en}}} |\mathbf{j}\cdot\nabla N_a|\right)^{-1} \tag{16}$$

$$\mathbf{j} = \frac{\nabla\rho^h}{\|\nabla\rho^h\|} \tag{17}$$

The parameter β is set as $\beta=1$ for smoother shocks and $\beta=2$ for sharper shocks.

In the 'YZU' version of the $YZ\beta$ shock-capturing, v_{SHOC} is defined as

$$v_{\mathrm{SHOC}} = \|\mathbf{Y}^{-1}\mathbf{Z}\|\left(\sum_{i=1}^{n_{\mathrm{sd}}}\left\|\mathbf{Y}^{-1}\frac{\partial\mathbf{U}^h}{\partial x_i}\right\|^2\right)^{\beta/2-1}\|\mathbf{Y}^{-1}\mathbf{U}^h\|^{1-\beta}\left(\frac{h_{\mathrm{SHOC}}}{2}\right)^{\beta} \tag{18}$$

In the 'YZ12' and 'YZU12' versions, v_{SHOC} is defined, based on the expressions given by Equations (12) and (18), by an averaging between the $\beta=1$ and $\beta=2$ selections

$$v_{\mathrm{SHOC}} = \tfrac{1}{2}((v_{\mathrm{SHOC}})_{\beta=1} + (v_{\mathrm{SHOC}})_{\beta=2}) \tag{19}$$

Remark 1

When the expressions given by Equations (12) and (18) were originally introduced in [9, 10], the intent was to have $\mathbf{Z}$ represent the residual and thus make v_{SHOC} residual-based. This point was made explicitly in [12, 13] by stating that the $YZ\beta$ shock-capturing parameters were 'based on scaled residuals'. This was the motivation behind the term $\|\mathbf{Y}^{-1}\mathbf{Z}\|$ in those two expressions. The selections given in [9, 10] for $\mathbf{Z}$ represent the steady-state and time-dependent versions of the residual for inviscid flows with no source or external-force terms. The terms with the exponents $\beta/2-1$ and β generate the correct local length scale. When $\beta=1$, from both expressions we obtain a definition for v_{SHOC} that has reduced sensitivity to how the scaling matrix $\mathbf{Y}$ is selected. But when $\beta=2$, it is with the term $\|\mathbf{Y}^{-1}\mathbf{U}^h\|^{1-\beta}$ in Equation (18) that the definition still has a reduced sensitivity to how $\mathbf{Y}$ is selected.

5. TEST COMPUTATIONS WITH STANDARD 2D TEST PROBLEMS

The test computations were carried out by using the V-SGS stabilization in combination with the YZ12 and YZU12 versions of the $YZ\beta$ shock-capturing and with the shock-capturing parameter introduced in [17], which is based on the δ_{91} parameter and the $\boldsymbol{\tau}_{\mathrm{VSGS}}$ matrix in terms of conservation variables that is now called $\delta_{91-\mathrm{MOD}}$. These three options are denoted, respectively, by 'VYZ12', 'VYZU12' and 'V91'. For the purpose of comparison, computations were carried out also by using the SUPG stabilization in combination with the YZ12 version, and that option is denoted by 'SYZ12'.

For completeness, we provide here the expression for the shock-capturing parameter proposed in [17]:

$$v_{\mathrm{SHOC}} = \delta_{91-\mathrm{MOD}} = max(0, \delta_{91} - \delta_{\tau}) \tag{20}$$

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where

$$\delta_{91} = \left\| \mathbf{A}_k^h \frac{\partial \mathbf{U}^h}{\partial x_k} \right\|_{\widetilde{\mathbf{A}}_0^{-1}} \Bigg/ \left(\sum_{j=1}^{n_{\mathrm{sd}}} \left\| \frac{\partial \xi_j}{\partial x_k} \frac{\partial \mathbf{U}^h}{\partial x_k} \right\|^2_{\widetilde{\mathbf{A}}_0^{-1}} \right)^{1/2} \tag{21}$$

$$\delta_{\tau} = \left\| \mathbf{A}_k \frac{\partial \mathbf{U}^h}{\partial x_k} \right\|_{\widetilde{\mathbf{A}}_0^{-1} \boldsymbol{\tau}\mathrm{VSGS}} \Bigg/ \left\| \frac{\partial \mathbf{U}^h}{\partial x_k} \right\|_{\widetilde{\mathbf{A}}_0^{-1}} \tag{22}$$

where ξ_j's are the element coordinates, and $\widetilde{\mathbf{A}}_0$ is the Jacobian of the transformation from the entropy variables to the conservation variables.

We compute three well-known, steady-state test problems: 'oblique shock', 'reflected shock' and 'parabolic bump'. These were used in many earlier publications, and here we compute them with meshes made of quadrilateral elements. In all three test computations, for the calculation of the scaling matrix $\mathbf{Y}$ given by Equation (13), we set the reference values of the components of $\mathbf{U}$ to 1.0 in computational units.

Oblique shock. Figure 1 shows the problem description. This is a Mach 2 uniform flow over a wedge at an angle of $-10°$ with the horizontal wall. The solution involves an oblique shock at an angle of 29.3° emanating from the leading edge. The computational domain is a square with $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. The inflow conditions are given as $M = 2.0$, $\rho = 1.0$, $u_1 = \cos 10°$, $u_2 = -\sin 10°$ and $p = 0.179$. This results in an exact solution with the following outflow data: $M = 1.64$, $\rho = 1.46$, $u_1 = 0.887$, $u_2 = 0.0$ and $p = 0.305$. All essential boundary conditions are imposed at the left and top boundaries, slip condition at the wall and no boundary conditions at the right boundary. The mesh is essentially uniform and consists of 20×20 elements. Figure 2 shows the density along $x = 0.9$, obtained with VYZ12, V91 and SYZ12. The solutions obtained with VYZ12 and SYZ12, which are indistinguishable from each other, clearly exhibit less dissipation than the solution obtained with V91. Figure 3 shows the solutions obtained with VYZ12 and VYZU12, which are indistinguishable from each other. The new shock-capturing parameters, in

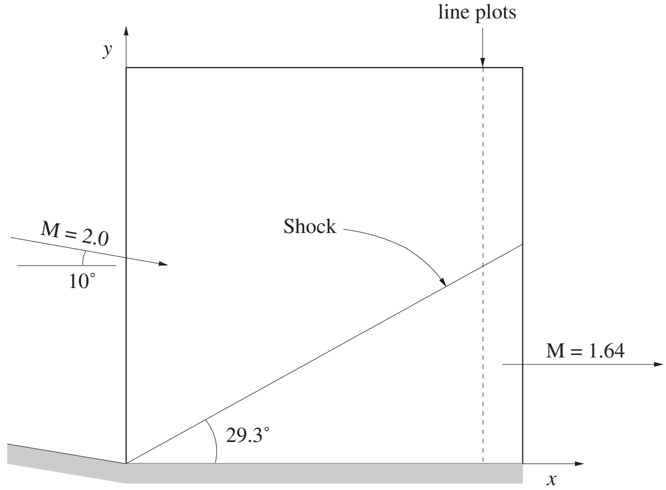

Figure 1. Oblique shock. Problem description.

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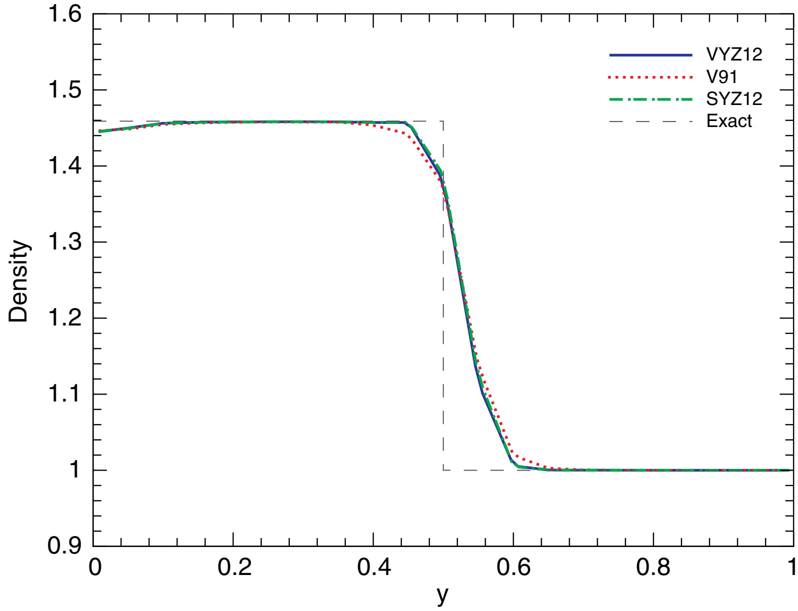

Figure 2. Oblique shock. Density along $x = 0.9$, obtained with VYZ12 (V-SGS and YZ12 combination), V91 (V-SGS and $\delta_{91-\mathrm{MOD}}$ combination) and SYZ12 (SUPG and YZ12 combination).

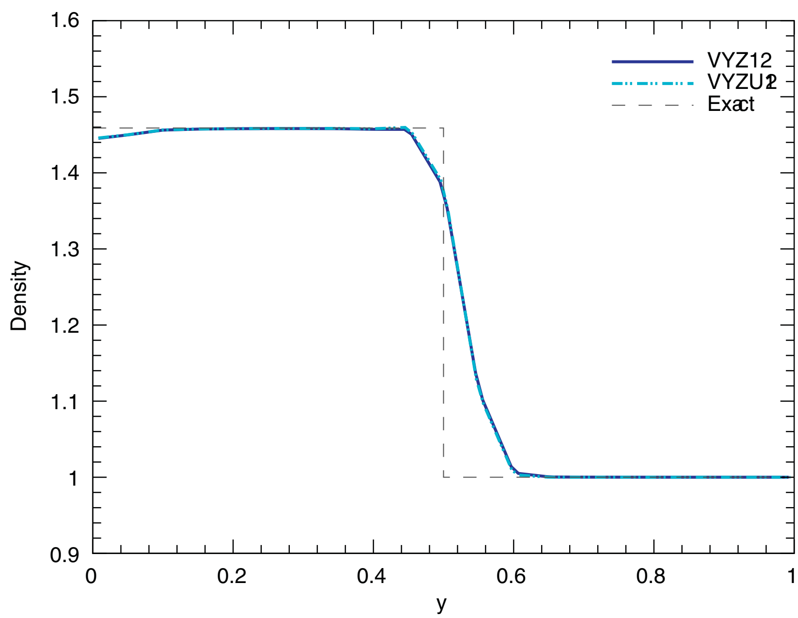

Figure 3. Oblique shock. Density along $x = 0.9$, obtained with VYZ12 (V-SGS and YZ12 combination) and VYZU12 (V-SGS and YZU12 combination).

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addition to being much simpler compared to $\delta_{91-\mathrm{MOD}}$, in computation of this test problem with the V-SGS stabilization they yield better shock quality than $\delta_{91-\mathrm{MOD}}$ does. This observation is essentially the same as the one reported in [11, 12] for computation of this test problem with the SUPG stabilization.

Reflected shock. Figure 4 shows the problem description. This problem consists of three flow regions (R1, R2 and R3) separated by an oblique shock and its reflection from the wall. The computational domain is a rectangle with $0\leqslant x\leqslant 4.1$ and $0\leqslant y\leqslant 1$. The inflow conditions in R1 are given as $M=2.9$, $\rho=1.0$, $u_1=2.9$, $u_2=0.0$ and $p=0.7143$. Specifying these conditions and requiring the incident shock to be at an angle of 29° results in an exact solution with the following flow data: R2: $M=2.378$, $\rho=1.7$, $u_1=2.619$, $u_2=-0.506$ and $p=1.528$; R3: $M=1.942$, $\rho=2.687$, $u_1=2.401$, $u_2=0.0$ and $p=2.934$. All essential boundary conditions are imposed at the left and top boundaries, slip condition at the wall, and no boundary conditions at the

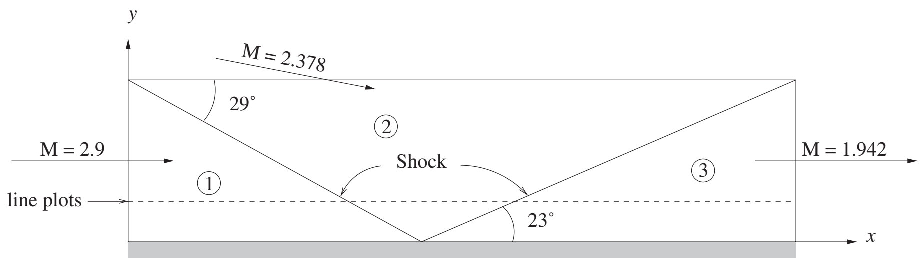

Figure 4. Reflected shock. Problem description.

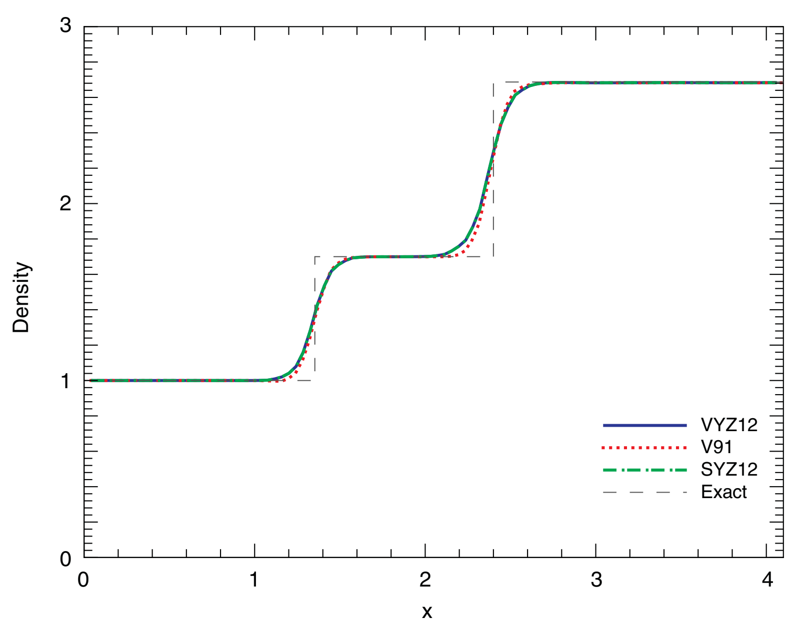

Figure 5. Reflected shock. Density along $y=0.25$, obtained with VYZ12 (V-SGS and YZ12 combination), V91 (V-SGS and $\delta_{91-\mathrm{MOD}}$ combination) and SYZ12 (SUPG and YZ12 combination).

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right boundary. The mesh is uniform and consists of 60×20 elements. Figure 5 shows the density along $y = 0.25$, obtained with VYZ12, V91 and SYZ12. The solutions obtained with VYZ12 and SYZ12, which are indistinguishable from each other, exhibit somewhat more dissipation than the solution obtained with V91. Figure 6 shows the solutions obtained with VYZ12 and VYZU12. In this case, the solution obtained with VYZU12 exhibits significantly less dissipation than the solution obtained with VYZ12. When we compare Figures 5 and 6, we conclude that the solution obtained with VYZU12 exhibits less dissipation than the solution obtained with V91. Also in this test problem, in computations with the V-SGS stabilization, the new, much simpler

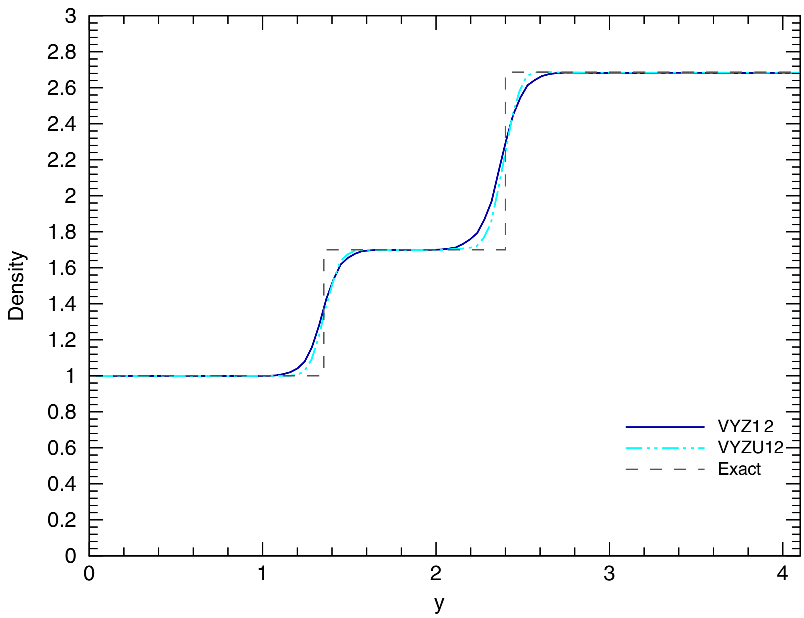

Figure 6. Reflected shock. Density along $y = 0.25$, obtained with VYZ12 (V-SGS and YZ12 combination) and VYZU12 (V-SGS and YZU12 combination).

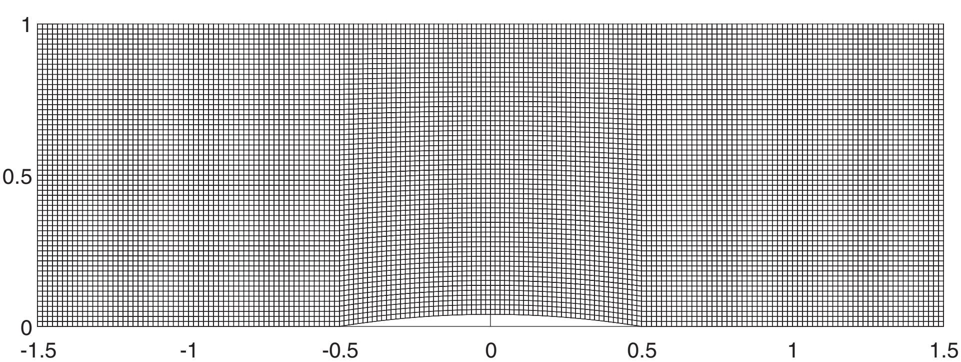

Figure 7. Parabolic bump. Problem geometry and mesh.

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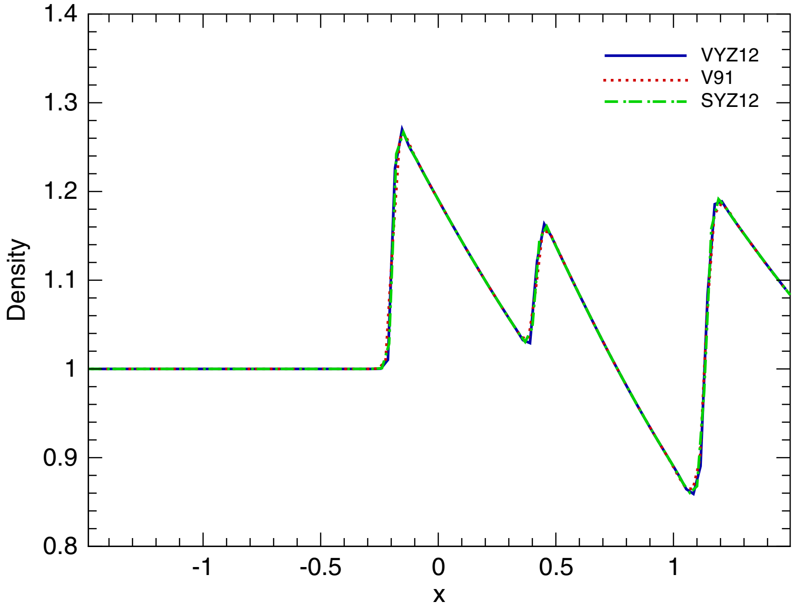

Figure 8. Parabolic bump. Density along $y = 0.5$, obtained with VYZ12 (V-SGS and YZ12 combination), V91 (V-SGS and $\delta_{91-\mathrm{MOD}}$ combination) and SYZ12 (SUPG and YZ12 combination).

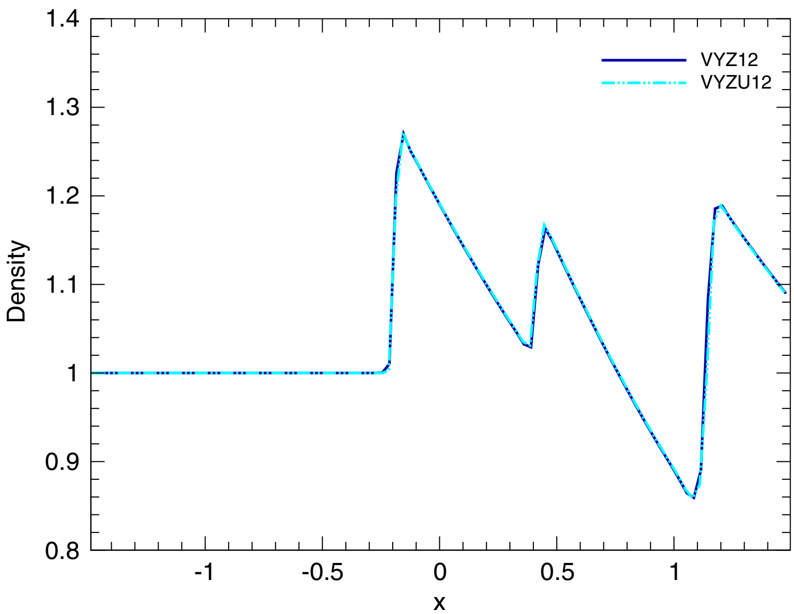

Figure 9. Parabolic bump. Density along $y = 0.5$, obtained with VYZ12 (V-SGS and YZ12 combination) and VYZU12 (V-SGS and YZU12 combination).

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shock-capturing parameters yield better shock quality than $\delta_{91-\text{MOD}}$ does. This is also essentially the same observation that was reported in [11, 12] for computation of this test problem with the SUPG stabilization.

Parabolic bump. Figure 7 shows the problem geometry and the mesh. The parabolic bump is symmetric, has a height of 0.05, starts at $x = -0.5$, and ends at $x = 0.5$. The inflow conditions (at the left boundary) and the conditions at the top boundary are given as $M = 1.4$, $\rho = 1.0$, $u_1 = 1.0$, $u_2 = 0.0$ and $p = 0.364$. Slip condition is imposed at the lower boundary and no boundary conditions at the right boundary. The mesh is almost uniform and consists of 184×60 elements. In the x-direction, there are 64 elements along the parabolic bump 60 elements before the bump and 60 after the bump. Figure 8 shows the density along $y = 0.5$, obtained with VYZ12, V91 and SYZ12. The solutions obtained with VYZ12 and SYZ12, which are indistinguishable from each other, exhibit slightly less dissipation than the solution obtained with V91. Figure 9 shows the solutions obtained with VYZ12 and VYZU12, which are essentially indistinguishable from each other.

6. CONCLUDING REMARKS

We described a finite element formulation of compressible flows in conservation variables, where the $YZ\beta$ shock-capturing technique is used in combination with the variable subgrid scale (V-SGS) method. The shock-capturing term provides additional stabilization near the shocks. How we define the shock-capturing parameter embedded in that term significantly influences the quality of the solution near the shocks. The $YZ\beta$ shock-capturing technique was introduced recently for use in combination with the streamline-upwind/Petrov–Galerkin (SUPG) formulation of compressible flows in conservation variables. It is simpler and less costly to compute with than the shock-capturing parameter δ_{91}, which was derived in 1991 from a shock-capturing parameter given for the entropy variables. The V-SGS method is based on an approximation of the class of SGS models derived from the Hughes variational multiscale (Hughes-VMS) method. We used three standard, 2D test problems to evaluate the performance of the V-SGS and $YZ\beta$ combination in computation of supersonic flows: 'oblique shock', 'reflected shock' and 'parabolic bump'. Compared to $\delta_{91-\text{MOD}}$, in addition to being much simpler, the $YZ\beta$ shock-capturing parameter yielded better shock quality in these test problems.

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